8.1 Proving the BAC - CAB rule

- The BAC-CAB rule states that for any three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$.

$$\vec{A} \times (\vec{B} \times \vec{C}) = B(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

To prove this identity, I'll use the summation convention with the Levi Civita symbol (antisymmetric tensor). The Levi-civita symbol ϵ_{ijk} is defined as

$$\epsilon_{ijk} \begin{cases} 1 & \text{if } (i,j,k) \text{ is a cyclic permutation } (123), (231) \text{ or } (312) \\ -1 & \text{if } (i,j,k) \text{ is an anticyclic permutation } (132), (321) \text{ or } (213) \\ 0 & \text{otherwise (when any two indices are equal)} \end{cases}$$

Starting with the left side of the identity, I can express the i-th component of $\vec{A} \times (\vec{B} \times \vec{C})$ as:

$$[\vec{A} \times (\vec{B} \times \vec{C})]_i = \epsilon_{ijk} A_j (\vec{B} \times \vec{C})_k$$

The cross product $\vec{B} \times \vec{C}$ can be written as:

$$(\vec{B} \times \vec{C})_k = \epsilon_{klm} B_l C_m$$

Substituting this into the previous equation:

$$[\vec{A} \times (\vec{B} \times \vec{C})]_i = \epsilon_{ijk} A_j \epsilon_{klm} B_l C_m$$

A key identity for products of Levi-Civita symbols is:

$$\epsilon_{ijk} \epsilon_{lm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

where δ_{ij} is the Kronecker delta, which equals 1 when $i = j$ and 0 otherwise. Using this identity:

$$[\vec{A} \times (\vec{B} \times \vec{C})]_i = A_j (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) B_l C_m$$

Simplifying with the properties of the Kronecker delta:

$$[\vec{A} \times (\vec{B} \times \vec{C})]_i = A_j \delta_{il} \delta_{jm} B_l C_m - A_j \delta_{im} \delta_{jl} B_l C_m$$

$$[\vec{A} \times (\vec{B} \times \vec{C}]_i = A_j B_i \delta_{jm} C_m - A_j C_i \delta_{jl} B_l$$

$$[\vec{A} \times (\vec{B} \times \vec{C})]_i = B_i (A_j C_j) - C_i (A_j B_j)$$

Recognizing that $A_j C_j = \vec{A} . \vec{C}$ and $A_j . B_j = \vec{A} . \vec{B}$. we get:

$$[\vec{A} \times (\vec{B} \times \vec{C})]_i = B_i (\vec{A} . \vec{C}) - C_i (\vec{A} . \vec{B})$$

In vector notation, this is:

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} . \vec{C}) - \vec{C}(\vec{A} . \vec{B})$$

A particularly important application of this identity is in vector calculus, specifically when dealing with the curl of a curl. If we let $\vec{A} = \nabla$, $\vec{B} = \nabla$, and $\vec{C} = \vec{E}$ (the electric field), we can derive:

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla . \vec{E}) - \nabla^2 \vec{E}$$

This identity is crucial in deriving the electromagnetic wave equation from Maxwell's equation. In vacuum, where $\nabla . \vec{E} = 0$ (no free charges), this simplifies to:

$$\nabla \times (\nabla \times \vec{E}) = -\nabla^2 \vec{E}$$

8.2 A parallel plate capacitor consists of two conducting plates separated by a distance d. When a potential difference V is applied, equal and opposite charges +Q and -Q accumulate on plates. I'll uses Guass's Law to find the electric field, then calculate the potential difference, and finally determine the capacitance:

Guass's Law in integral form states:

$$\oint_S \vec{D} \cdot d\vec{A} = Q_{enclosed}$$

where $\vec{D}$ is electric displacement field, $d\vec{A}$ is the differential area element, and $Q_{enclosed}$ is charge enclosed by Guassian surface

For my Guassion surface, I'll choose a rectangular box that passes through one of the plates, with one face inside the capacitor and the other outside. By symmetry, the electric field must be perpendicular to the plates and uniform in the region between them.

Inside the capacitor, the field is constant, so:

$$\int_{BCHIB} \vec{D} \cdot d\vec{A} = \int_{ACHIA} \vec{D} \cdot d\vec{A}$$

Outside the capacitor, the field is zero (as we'll verify), so:

$$\int_{BEFIB} \vec{D} \cdot d\vec{A} = 0$$

The field is the same on both sides of the capacitor:

$$\vec{D} = 0 \text{ at infinity}$$

When the Gaussian surface encloses charge, we have:

$$\int_{BCHIB} \vec{D} \cdot d\vec{A} = \int_{BCHIB} \rho \, dV$$

For a uniform charge distribution on the plate:

$$\epsilon E A = Q$$

Where ϵ is the permittivity of the medium (vacuum in this case), E is the electric field magnitude, and A is the area of the plate

Solving the electric field:

$$E = \frac{Q}{\epsilon A}$$

This shows that the electric field between the plates is uniform & perpendicular to the plates, with magnitude proportional to the charge density

The potential difference between the plates is ~~uniform and perpendicular~~ ~~to the plates~~ related to the electric field by:

$$V = -\int \vec{E} \cdot d\vec{l}$$

Since the field is uniform & perpendicular to the plates, and the path of integration is along the field lines:

$$V = Ed = \frac{Qd}{\epsilon A}$$

The capacitance is defined as the ratio of charge to potential difference:

$$C = \frac{Q}{V}$$

Substituting our expression for V:

$$C = \frac{Q}{\frac{Qd}{\epsilon A}} = \frac{\epsilon A}{d}$$

Therefore, the capacitance of a parallel plate capacitor is:

$$C = \frac{\epsilon A}{d}$$

b) Maxwell's modification to Ampère's law includes a term for the displacement current:

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The displacement current density is defined as:

$$\vec{J}_D = \epsilon_0 \frac{\partial E}{\partial t}$$

For a capacitor with time-varying fields, the displacement current through a surface cutting across the capacitor is:

$$\int_{CH} \frac{\partial \vec{D}}{\partial t} \cdot d\vec{A} = \int_{CH} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{A}$$

Since the electric field is uniform b/w the plates and zero elsewhere and pointing in the direction of the surface normal:

$$\int_{CH} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{A} = \frac{\partial E}{\partial t} A$$

The electric field is related to the potential difference by $E = V/d$, so:

$$\frac{\partial E}{\partial t} A = \frac{1}{d} \frac{dV}{dt} A$$

The external current is related to the rate of change of charge on the capacitor:

$$I = \frac{dQ}{dt}$$

Using the relation $Q = CV = \epsilon AV/d$:

$$I = \frac{d}{dt}\left(\frac{\epsilon AV}{d}\right) = \frac{\epsilon A}{d} \frac{dV}{dt}$$

Therefore:

$$\int_{CH} \frac{\partial \vec{D}}{\partial t} \cdot d\vec{A} = \epsilon \frac{\partial E}{\partial t} A = \epsilon \frac{1}{d} \frac{dV}{dt} A = I$$

c) The energy stored in an electric field is given by the volume integral of the energy density:

$$U = \frac{1}{2} \int_V \vec{D} \cdot \vec{E} \, dV$$

For a linear dielectric, $\vec{D} = \epsilon \vec{E}$, so:

$$U = \frac{1}{2} \int_V \epsilon E^2 dV$$

In a parallel plate capacitor, the electric field is uniform between the plates and zero elsewhere. The volume of the field region is Ad, where A is the plate area and d is the separation. Therefore:

$$U = \frac{1}{2} \epsilon E^2 A d$$

We know that $E = Q/(\epsilon A)$, so:

$$U = \frac{1}{2} \epsilon \left(\frac{Q}{\epsilon A} \right)^2 Ad = \frac{1}{2} \frac{Q^2}{\epsilon A^2} \epsilon A d = \frac{1}{2} \frac{Q^2}{\epsilon A} d$$

Using $C = \epsilon A/d$ and $V = Qd/(\epsilon A)$:

$$U = \frac{1}{2} \frac{Q^2}{\epsilon A} d = \frac{1}{2} \frac{Q^2}{\epsilon A/d} = \frac{1}{2} \frac{Q^2}{C}$$

Alternatively, using $Q = CV$:

$$U = \frac{1}{2} \frac{(CV)^2}{C} = \frac{1}{2} CV^2$$

This gives us the familiar formula for energy stored in a capacitor:

$$U = \frac{1}{2} CV^2$$

d) First, I'll calculate energy stored in the battery:

Energy = Voltage × Current × Time = $10V \times 10A \times 3600s = 3.6 \times 10^5 J$

For a capacitor to store this energy at 10 V, using $U = \frac{1}{2} CU^2$:

$$3.6 \times 10^5 J = \frac{1}{2} C (10V)^2$$

Solving for C:

$$C = \frac{2 \times 3.6 \times 10^5 J}{(10V)^2} = \frac{7.2 \times 10^5 J}{100V^2} = 7200F$$

This is an extremely large capacitance for a conventional capacitor.

Using the formula for a parallel plate capacitor with vacuum dielectric ($\epsilon_0 = 8.854 \times 10^{-12}$ F/m) and plate spacing $d = 10^{-6}m = 1\mu m$:

$$C = \frac{\epsilon_0 A}{d}$$

Solving for the area A:

$$A = \frac{Cd}{\epsilon_0} = \frac{7200F \times 10^{-6}m}{8.854 \times 10^{-12} F/m} = 8 \times 10^8 m^2$$

If the plates are 10cm × 10cm ($0.1m \times 0.1m = 0.01m^2$) the number of plates needed would be:

$$\text{No. of plates} = \frac{8 \times 10^8 m^2}{0.01m^2} = 8 \times 10^{10}$$

With a spacing of $1\mu m$ b/w plates, the stack height would be:

$$\text{Height} = 8 \times 10^{10} \times 10^{-6} m = 8 \times 10^4 m = 80 km$$

This is impractical for a conventional capacitor design.

Supracapacitors can achieve much higher capacitances than conventional capacitors by replacing the dielectric with an electrolyte containing ions that can move to create a double layer around an insulating barrier.

8.3 I'll use Ampère's Law, which in integral form is:

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed}$$

where $\vec{B}$ is the magnetic field, $d\vec{l}$ is a differential element of the integration path, μ_0 is the permeability of free space, and $I_{enclosed}$ is the current enclosed by the integration path.

Consider a rectangular integration path as shown in fig A1.9, with one side inside the solenoid and parallel to its axis, and the other side outside.

For the side inside the solenoid, the contribution to the line integral is:

$$\int_{inside} \vec{B} \cdot d\vec{l} = B_{inside} \cdot L$$

where L is length of the path inside the solenoid, and B_{inside} is the magnitude of the magnetic field inside.

The current enclosed by our integration path is the current per turn times the number of turns closed:

$$I_{enclosed} = I \cdot (n \cdot L)$$

where n = number of turns per unit length, L is length of our path inside solenoid. Applying Ampère's Law:

$$B_{inside} \cdot L = \mu_0 I n L$$

Solving for B inside:

$$B_{inside} = \mu_0 n I$$

Therefore, the magnetic field inside an infinite solenoid is:

$$\vec{B} = \mu_0 n I \hat{z}$$

where $\hat{z}$ is unit vector along the axis of the solenoid.

b) The energy density in a magnetic field is given by:

$$u = \frac{1}{2}\vec{B}\cdot\vec{H}$$

For a linear magnetic material, $\vec{B} = \mu\vec{H}$, where μ is permeability. In vacuum or air, $\mu = \mu_0$ and the energy density becomes:

$$u = \frac{1}{2}\frac{B^2}{\mu_0}$$

The total energy stored in the magnetic field is the volume integral of the energy density:

$$U = \int_V u\,dV = \frac{1}{2}\int_V \vec{B}\cdot\vec{H}\,dV$$

For a solenoid, the magnetic field is uniform inside and zero outside, as we found in part (a), Inside the solenoid, $\vec{B} = \mu_0 n I\hat{z}$ and $\vec{H} = nI\hat{z}$.

The volume of the solenoid is $V = \pi r^2 l$, where r is radius and l is length. Since the field is uniform inside, the energy integral simplifies to:

$$U = \frac{1}{2}\vec{B}\cdot\vec{H}\cdot V = \frac{1}{2}\mu_0 n I\cdot nI\cdot\pi r^2 l$$

Simplifying:

$$U = \frac{1}{2}\mu_0 n^2 I^2 \pi r^2 l$$

The induction of a solenoid is defined as the ratio of the magnetic flux linkage to the current:

$$L = \frac{N\phi}{I}$$

where N is total no. of turns $(N = nl)$ and ϕ is the magnetic flux through each turn. The magnetic flux through each turn is: $\phi = B\cdot\pi r^2 = \mu_0 n I\cdot\pi r^2$

Therefore, the inductance is:

$$L = \frac{nl \cdot \mu_0 n I \cdot \pi r^2}{I} = \mu_0 n^2 l \pi r^2$$

we can now express the energy in terms of the inductance:

$$U = \frac{1}{2}\mu_0 n^2 I^2 \pi r^2 l = \frac{1}{2} L I^2$$

(c)

$$\vec{F} = -\nabla U$$

In our case, we're interested in radial force, which is:

$$F_r = \frac{\partial U}{\partial r}$$

From part (b), we know that the energy stored in solenoid is:

$$U = \frac{1}{2}\mu_0 n^2 I^2 \pi r^2 l$$

Taking the derivative with respect to r:

$$F = -\frac{\partial U}{} = -\frac{1}{}\mu_0 n^2 I^2 \pi . 2r . l = -\mu_0 n^2 I^2 \pi r l$$

For a constant current, the force is:

$$F_r = \frac{\partial U}{\partial r} = \mu_0 n^2 I^2 \pi r l$$

We can express this in terms of the magnetic field $B = \mu_0 n I$:

$$F_r = \frac{B^2}{\mu_0} \pi r l$$

For the 10T MRI magnet with bore diameter 1m (radius r = 0.5m) and length l = 2m:

$$F_r = \frac{(10T)^2}{4\pi \times 10^{-7} H/m} \pi \times 0.5m \times 2m$$

$$F_r = \frac{100T^2 \times \pi \times 0.5m \times 2m}{4\pi \times 10^{-7} H/m}$$

$$F_r = \frac{100T^2 \times \pi \times 1m^2}{4\pi \times 10^{-7} H/m} = \frac{100\pi \; T^2 m^2}{4\pi \times 10^{-7} \; H/m}$$

$$F_T = \frac{25 T^2 m^2}{10^{-7} H/m} = 2.5 \times 10^8 N$$

$F_T = 2.5 \times 10^8 N \approx 2.5 \times 10^7 kg \times 9.8 m/s^2 \approx 2.5 \times 10^7 kg \approx 25,000$ tons

$\boxed{8.4}$ First, I need to find the magnetic field produced by a straight wire carrying current I. Using Ampere's Law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed}$$

For a circular path of radius r around the wire, the magnetic field must be tangential to the circle by symmetry, and its magnitude must be constant at all points on the circle. Therefore:

$$B \cdot 2\pi r = \mu_0 I$$

Solving for B:

$$B = \frac{\mu_0 I}{2\pi r}$$

Now, consider two parallel ~~lines~~ wires carrying currents I_1 and I_2, separated by a distance r. The magnetic field from the first wire at the location of the second wire is:

$$B_1 = \frac{\mu_0 I_1}{2\pi r}$$

The force on a current element in the second wire is given by:

$$d\vec{F} = I_2 \, d\vec{l} \times \vec{B_1}$$

For a length L of the second wire:

$$F = I_2 L B_1 = I_2 L \frac{\mu_0 I_1}{2\pi r} = \frac{\mu_0 I_1 I_2 L}{2\pi r}$$

For the specific case in the definition, where

$I_1 = I_2 = 1A$ and $r = 1m$:

$$F = \frac{\mu_0 \cdot 1A \cdot 1A \cdot L}{2\pi r \cdot 1m} = \frac{\mu_0 L}{2\pi}$$

Substituting $\mu_0 = 4\pi \times 10^{-7}\ H/m$:

$$F = \frac{4\pi \times 10^{-7}\ H/m \cdot L}{2\pi} = 2 \times 10^{-7}\ N/m \cdot L$$

8.5 a) The force on a current element in a magnetic field is given by:

$$d\vec{F} = I\,d\vec{l} \times \vec{B}$$

For a complete circuit (like a coil), the total force is the line integral around the circuit:

$$\vec{F} = I \oint d\vec{l} \times \vec{B}$$

This can be rewritten as:

$$\vec{F} = -I \oint \vec{B} \times d\vec{l}$$

For the Kibble balance, we're interested in the vertical component of the force, which balances the weight of the test mass. The vertical component is:

$$F_z = I \oint (B_y dx - B_x dy)$$

Where B_x and B_y are the horizontal components of the magnetic field, and the integration is performed around the coil.

Green's theorem allows us to convert this line integral to an area integral:

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Applying this to our force equation with $P = -B_x$ and $Q = B_y$:

$$F_z = I \iint \left(\frac{\partial B_y}{\partial x} - \frac{\partial(-B_x)}{\partial y}\right) dx\,dy = I \iint \left(\frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y}\right) dx\,dy$$

The divergence of the magnetic field in three dimensions is:

$$\nabla \cdot B = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

A fundamental property of magnetic fields is that they are divergence - free:

$$\nabla \cdot \vec{B} = 0$$

This means:

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = -\frac{\partial B_z}{\partial z}$$

Substituting this into our force equation:

$$F_z = I \iint \left(-\frac{\partial B_z}{\partial z}\right) dx dy = -I \iint \frac{\partial B_z}{\partial z} dx dy$$

If the vertical gradient of the magnetic field $(\partial B_z / \partial z)$ is approximately constant over the area of the coil, we can simplify this to:

$$F_z = -I \frac{\partial B_z}{\partial z} \iint dx dy = -I \frac{\partial B_z}{\partial z} A$$

Where A is the area of the coil.

In the static phase of the Kibble balance, this electro-magnetic force balances the gravitational force on the test mass:

$$mg = -I \frac{\partial \phi}{\partial z}$$

where $\Phi = B_z A$ is the magnetic flux through the coil.

(b) When a conductor moves through a magnetic field, an electromotive force is induced according to Faraday's Law. The voltage around a closed loop is:

$$V = \oint \vec{E} \cdot d\vec{l}$$

Using Stokes theorem, this can be related to the curl of the electric field:

$$V = \int_S (\nabla \times \vec{E}) \cdot d\vec{A}$$

From Maxwell's Equations, the curl of the electric field is related to the time derivative of the magnetic field.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Therefore:

$$V = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A} = -\frac{d\phi}{dt}$$

where ϕ is the magnetic flux through the coil.

When the coil moves vertically with velocity V_z, the rate of change of flux can be expressed using the chain rule:

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial z} \frac{dz}{dt} = \frac{\partial \phi}{\partial z} V_z$$

Therefore, the induced voltage is:

$$V = -\frac{d\phi}{dt} = -\frac{\partial \phi}{\partial z} V_z$$

c) From the static phase, we have:

$$mg = -I \frac{\partial \phi}{\partial z}$$

From the dynamic phase, we have:

$$V = -\frac{\partial \phi}{\partial z} V_z$$

Dividing the first equation by the second.

$$\frac{mg}{V} = \frac{I}{\frac{\partial \phi}{\partial t}} \cdot \frac{\frac{\partial \phi}{\partial z}}{V_z} = \frac{I}{V_z}$$

Solving for the mass:

$$m = \frac{IV}{gV_z}$$

8.6

a) The power transported by electromagnetic radiation is described by the Poynting vector:

$$\vec{P} = \vec{E} \times \vec{H}$$

where $\vec{E}$ is the electric field and $\vec{H}$ is the magnetic field

For a plane wave in vacuum, the magnitude of the Poynting vector is:

$$|\vec{P}| = \sqrt{\frac{\epsilon_0}{\mu_0}} E^2$$

where ϵ_0 is the permitivity of free space and μ_0 is the permeability of free space.

For a sinusoidal electromagnetic wave, the electric field can be written as:

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

The physical electric field is the real part of this expression.

Since $|e^{i\vec{k}\cdot\vec{x}}| = 1$, and the time average of $[\cos(\omega t)]^2$ is $1/2$, the time-averaged Poynting vector magnitude is:

$$\langle |P| \rangle = \frac{1}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} E_0^2$$

Given a power density of $1\,kW/m^2 = 10^3 W/m^2$, we can solve for the electric field amplitude:

$$10^3 W/m^2 = \frac{1}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} E_0^2$$

Rearranging to solve for E_0:

$$E_0^2 = \frac{2 \times 10^3 W/m^2}{\sqrt{\frac{\epsilon_0}{\mu_0}}}$$

The quantity $\sqrt{\epsilon_0/\mu_0}$ is the reciprocal of the impedance of free space, which is approximately $1/377^1$.

Therefore:

$$E_0^2 = 2\times10^3\,W/m^2 \times 377 \simeq 7.54\times10^5\,V^2/m^2$$

Taking the square root:

$$E_0 \simeq 868\,V/m$$

(b) For a 1 W laser focused to a $1mm^2$ spot ($10^{-3}m \times 10^{-3}m$), the power density is:

$$\text{Power density} = \frac{1W}{10^{-6}m^2} = 10^6\,W/m^2$$

Using the same relationship as before:

$$E_0^2 = \frac{2\times10^6\,W/m^2}{\sqrt{\frac{\varepsilon_0}{\mu_0}}} = 2\times10^6\,W/m^2 \times 377 \simeq 7.54\times10^8\,V^2/m^2$$

Taking square root:

$$E_0 \simeq 2.7\times10^4\,V/m$$

The diffraction limit for focusing light is approximately the wavelength of the light. For visible light, this is on the order of 1 m. For a 1 W laser focused to a $1m^2$ spot ($10m \times 10m$), the power density is:

$$\text{Power density} = \frac{1W}{10^{-12}m^2} = 10^{12}\,W/m^2$$

Using the same relationship:

$$E_0^2 = \frac{2\times10^{12}\,W/m^2}{\sqrt{\frac{\varepsilon_0}{\mu_0}}} = 2\times10^{12}\,W/m^2\times377 \simeq 7.54\times10^{14}\,V^2/m^2$$

Taking the square root:

$$E_0 \simeq 2.7\times10^7\,V/m$$